Recitation 6

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Review

An $m \times n$ matrix A defines $A: \mathbb{R}^n \to \mathbb{R}^m$. The following are equivalent:

- A is **onto**;
- columns of A span $\mathbb{R}^m$;
- system $Ax = b$ has a solution (is consistent) for all b;
- **every row** of A has a pivotal position;

Once again, TFAE:

- A is **one-to-one**;
- $Ax = Ay$ implies that $x = y$ (i.e. A maps different vectors to different vectors);
- $Ax = 0$ has only the trivial solution (i.e. A doesn't kill any non-zero vectors);
- columns of A are linearly independent;
- **every column** of A is pivotal.

Basis: vectors $v_1, \ldots, v_n$ in a vector space V for a **basis** if and only if

- $v_1, \ldots, v_n$ are independent;
- $v_1, \ldots, v_n$ span the whole V.

Basis for $Col(A)$: pivotal columns of the original matrix A form a basis of $Col(A)$ (but **not!** the columns of the reduced form of A).
Basis for $Span(v_1, \ldots, v_n)$: it's the same as asking for a basis of $Col(A)$ with $A = (v_1 \ldots v_n)$.
Basis for $Nul(A)$: you need to solve $Ax = 0$. So do that. Some variables will be free, some not. Express the solution as a vector, substituting non-free variables in terms of free ones. Then plug in 1 for one of free variables, put rest 0. This gives a vector. Do that for all free variables, get a bunch of vectors. This would be a basis for $Nul(A)$.

Coordinates relative to a basis: Suppose $v_1, \ldots, v_n$ is a basis of V, and b is any vector. To find **coordinates** of b you need to find scalars $x_1, \ldots, x_n$ such that $x_1 v_1 + \cdots + x_n v_n = b$. So really you **need to solve system of equations** $Ax = b$ with $A = (v_1 \ldots v_n)$ (i.e. vectors v_i are columns of A).

Dimension: if V is a vector space, dimension is the number of vectors in **any basis**. So to find dimension, find a basis. For that, find vectors $v_1, \ldots, v_n$ that span V, and only keep linearly independent.
For $Col(A)$ and $Nul(A)$: $\dim Nul(A) =$ the number of free variables in $Ax = 0$. $\dim Col(A) =$ the number of pivot columns of A.

Rank: the rank of a matrix A is the $\dim Col(A)$, which equals to the number of pivot columns of A.
The rank theorem: is A is $m \times n$, i.e. A defines a transformation $\mathbf{R}^n \to \mathbf{R}^m$, then
$\mathbf{rank(A)} + \dim \mathbf{Nul(A)} = \mathbf{n}$.

Problems

Problem 1. Let $\mathcal{B} = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ be two vectors in $\mathbb{R}^2$. Prove that they form a basis in $\mathbb{R}^2$. Let $x = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. Find the coordinates $[x]_\mathcal{B}$ of the vector x relative to the basis $\mathcal{B}$. Use the inverse matrix to do that.

Problem 2. Do polynomials $\{1+t, t^2-1, t+t^2-1\}$ form a basis of $\mathbb{P}_2$? Find the coordinates of $3t^2$ relative to that basis.

Problem 3. Let $V \subset \mathbb{R}^4$ be a subspace of vectors $[a,b,c,d]^T$ satisfying $a+b+c = d-2c = 0$. What is the dimension of that space? (Hint: V is a null space of some matrix.) Find a basis of V.

Problem 4. The dimension of the null space of a 5×9 matrix A is 8. What is the dimension of $Col(A)$?

Problem 5. Suppose a person named X tells you that he has a 5×8 matrix B with null space being 2-dimensional. Why is this person a dirty liar?

Problem 6. Suppose another person, named Y, tells you that for any square matrix A, the column space $Col(A)$ is the same as the row space $Row(A)$. Is he any better than the person X?

Problem 7. Suppose A is an $m \times n$ matrix. Which of the spaces $Row(A), Col(A), Nul(A), Row(A^T), Col(A^T), Nul(A^T)$ are in $\mathbb{R}^n$, and which are in $\mathbb{R}^m$?

Problems from the past (kind of)

Problem 8. Determine all values of a such that the vectors $\begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ a+6 \end{bmatrix}$ are linearly independent.

Problem 9. Suppose you know that the **augmented matrix** of a system $Ax = b$ after some row reduction looks like

$$\begin{bmatrix} 1 & 3 & -2 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

. You don't know the original matrix A. Find solutions of the system $Ax = b$.

Suppose you know that the first column of the original matrix A is $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. What is the second column of A?

Find the rank of A.

Problem 10. Is the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$ invertible?

How can you use inverse of A to solve the system of equations

$$\begin{cases} x_1 - 2x_2 + x_3 = 7 \\ 2x_1 - 7x_2 + 3x_3 = 3 \\ -2x_1 + 6x_2 - 4x_3 = 0 \end{cases}$$

Find the inverse and solve the system.

Problem 11. Is vector $b = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ in the columns space of the matrix $C = \begin{bmatrix} 5 & 1 \\ -5 & 1 \\ 4 & 2 \end{bmatrix}$?